ELSEVIER

Available online at www.sciencedirect.com

ScienceDirect

Procedia Social and Behavioral Sciences 2 (2010) 7595–7596

Procedia
Social and Behavioral Sciences

Sixth International Conference on Sensitivity Analysis of Model Output

Distributional sensitivity analysis

Douglas L. Allaire[a,]* and Karen E. Willcox[a]

[a]*Department of Aeronautics and Astronautics, Massachusetts Institute of Technology, Cambridge, MA, 02139, USA*

Abstract

Among the uses for global sensitivity analysis is factor prioritization. A key assumption for this is that a given factor can, through further research, be fixed to some point on its domain. For factors containing epistemic uncertainty, this is an optimistic assumption, which can lead to inappropriate resource allocation. Thus, this research develops an original method, referred to as distributional sensitivity analysis, that considers which factors would on average cause the greatest reduction in output variance, given that the portion of a particular factor□s variance that can be reduced is a random variable. A key aspect of the method is that the analysis is performed directly on the samples that were generated during a global sensitivity analysis using acceptance/rejection sampling. In general, if for each factor, N model runs are required for a global sensitivity analysis, then those same N model runs are sufficient for a distributional sensitivity analysis.

Keywords: distributional sensitivity analysis; global sensitivity analysis; factor prioritization; variance reduction

1. Main text

Consider a model, $Y = f(\mathbf{x})$, where $\mathbf{x} = [X_1,...,X_m]^T$ and $X_1,...,X_m$ are random variables and thus, Y is a random variable as well. As shown in Saltelli et al. (2008), the main effect sensitivity index for factor *i* of this model is given as, $S_i = \text{var}(E[Y|X_i])/\text{var}(Y)$, which by the definition of total variance, is related to the expected value of the variance of the output through $E[\text{var}(Y|X_i)] = \text{var}(Y)$ □ $S_i\text{var}(Y)$. Let X_i^o be the random variable defined by the original distribution for some factor *i,* and X_i' be the random variable defined by a new distribution for factor *i* after some further research has been done. These variables have corresponding main effect sensitivity indices S_i^o and S_i' respectively. Then we can define the ratio of the variance of factor *i* that cannot be reduced and the total variance of the original distribution of factor *i* as $\delta = \text{var}(X_i')/\text{var}(X_i^o)$. Assuming further research reduces the variance of factor *i,* $\delta \in [0,1]$. Since it cannot be known in advance how much variance reduction for a given factor is possible through further research, the distributional sensitivity analysis method casts δ as a uniform random variable, Δ, on [0,1], in keeping with the principle of maximum uncertainty (Ayyub, B. & Klir, G. 2006).

Given that the variance of factor *i* that may be reduced is a random percentage, 100(1-Δ)%, of the total original variance of factor *i,* a distributional sensitivity index function can be defined as $\text{adj}S_i(\delta) = \text{var}(Y^o)S_i^o$ □ $E[\text{var}(Y')S_i'|\Delta=\delta]/\text{var}(Y^o)$, where $\text{adj}S_i$ is to be read as, □the adjusted main effect sensitivity index of factor i,□S_i^o is

* Corresponding author. Tel.: +1-617-320-2394. *E-mail address*: dallaire@mit.edu.

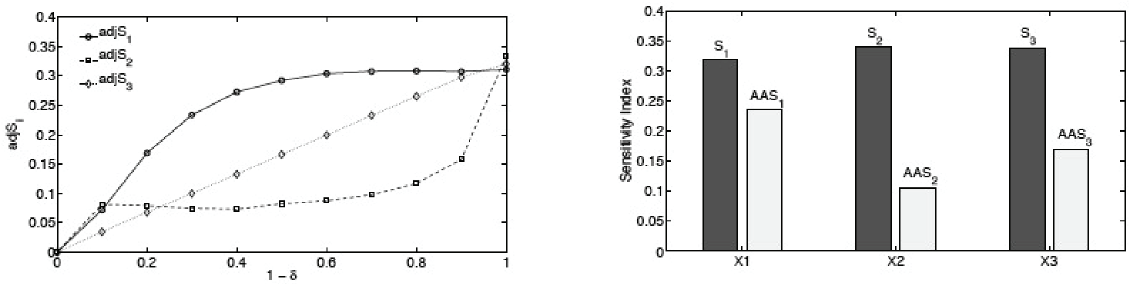
doi:10.1016/j.sbspro.2010.05.134

the original main effect sensitivity index of factor *i,* and E[var(Y')S_i'|Δ=δ] is the expected value of the product of the variance of the output and the main effect global sensitivity index of factor *i* taken over all *reasonable* distributions with 100δ% of the variance of the original distribution. Reasonable distributions are defined by the analyst.

The adjusted main effect sensitivity index function is interpreted as the main effect sensitivity index for factor *i* if it is known that only 100(1-δ)% of the factor's variance can be reduced. This can be seen by noting that var(Y^o) S_i^o is the expected value of the variance of Y^o that is due to factor *i*, and var(Y')S_i' is the expected value of the variance of Y' that is due to factor *i* after 100(1-δ)% of factor *i*'s variance has been reduced. Since there are many ways to reduce the variance of factor *i* by 100(1-δ)%, the expected value of var(Y')S_i' is taken over all reasonable distributions for which 100(1-δ)% has been reduced. Thus, var(Y^o)S_i^o - E[var(Y')S_i'|Δ=δ] is the amount of variance in Y^o that cannot be reduced further if factor *i*'s variance can only be reduced by 100(1-δ)%.

If it is assumed that all of the variance of a given factor can be reduced, then δ = 0, and for a given factor *i,* this means that E[var(Y')S_i'|Δ=0] = 0, since once all of the variance of factor *i* has been reduced, factor *i* will simply become a constant, and thus, S_i' = 0. Therefore, when δ = 0, adjS_i(0) = S_i^o, and distributional sensitivity analysis reduces to global sensitivity analysis. However, the value δ takes will generally be unknown and is considered to be a uniform random variable, Δ, on the interval [0,1]. Thus, the expected value of adjS_i(Δ) can be taken to give an *average adjusted main effect sensitivity index* (AAS) for each factor as AAS_i = E[adjS_i(Δ)]. The average adjusted main effect sensitivity index for each factor is then an index that can be used to quantitatively rank factors based on the average amount of output variance that can be reduced when further research in done on a particular factor.

The following example reveals the additional insight gained by using distributional sensitivity analysis for factor prioritization. Consider a model given by $f(X_1,X_2,X_3) = 0.1\exp(X_1) + 20\exp(-X_2) + 11X_3$, where X_1~T(0,6,1/2), X_2~ T(0,6,1/2), X_3~T(0,2,1), and ~T(α,β,γ) represents a triangular distribution with minimum, α, maximum, β, and mode γ. Figure (a) presents the adjusted main effect sensitivity indices of each factor for values of δ = 0.0, 0.1,..., 1.0. The figure shows that factor prioritization depends on the variance that is assumed reducible for each factor. Figure (b) compares the main effect sensitivity indices estimated by global sensitivity analysis for each factor with the average adjusted main effect sensitivity indices estimated via distributional sensitivity analysis. The global sensitivity results suggest that the ranking for factor prioritization be factor 2, 3, and then 1. However, the distributional sensitivity results suggest that the ranking be factor 1, 3, and then 2, with clear differences in index magnitudes. In this case, assuming the variance of a given factor can be reduced to zero through further research leads to a different conclusion regarding which factors should be researched than distributional sensitivity analysis, which assumes the amount of variance that can be reduced for a given factor is a random variable. Since the notion that all of the variance of a given factor can be reduced through further research is optimistic, the distributional sensitivity analysis results are considered more reliable and are recommended for use in favor of global sensitivity analysis for factor prioritization.



(a) Adjusted Main Effect Sensitivity Indices

(b) Average Adjusted Main Effect Sensitivity Indices and Main Effect Sensitivity Indices

2.

Saltelli A., Ratto M., Andres T., Campolongo F., Cariboni J., Gatelli D., Saisana M., and Tarantola S., 2008: *Global Sensitivity Analysis: The Primer*, John Wiley & Sons, Ltd., West Sussex England.

Ayyub B. and Klir G. 2006: *Uncertainty Modeling and Analysis in Engineering and the Sciences,* Taylor & Francis Group, Boca Raton, Florida, USA.